Math 524 Exam 3 Solutions

Problems 1-4 are for the vector space $\mathbb{R}_2[t]$, real polynomials of degree at most 2. We define $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ via $L(f) = (t-1)\frac{df}{dt}$.

1. Directly calculate $[L]_E$, for the basis $E = \{1, t, t^2\}$.

   We again apply L to each basis element: $L(1) = 0, L(t) = t-1, L(t^2) = 2t^2 - 2t$. Hence $[L]_E = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$.

2. Directly calculate $[L]_B$, for the basis $B = \{1, t-1, (t-1)^2\}$.

   We apply L to each basis element: $L(1) = 0, L(t-1) = t-1, L((t-1)^2) = 2(t-1)^2$. These are very easy to express as linear combinations of B, hence $[L]_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Note that this basis is more natural for dealing with L.

3. Calculate P_{BE}, P_{EB}, and demonstrate the relationship between them and $[L]_B, [L]_E$.

   It is simpler to begin with $P_{EB} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$, then $P_{BE} = P_{EB}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. There are several ways to express the mutual relationship; for example $[L]_B = P_{BE}[L]_E P_{EB}$. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

4. Find a basis for the kernel of L. Find a basis for the range of L.

   Looking at $[L]_B$, we can see that the kernel is 1-dimensional and the range is 2-dimensional. A basis for the kernel will consist of a single element that L sends to 0, for example $\{1\}$. A basis for the range will consist of two linearly independent elements from the range, for example $\{t-1, 2t^2 - 2t\}$.

5. Consider the operator $M : \mathbb{R}[t] \to \mathbb{R}[t]$, an operator on real polynomials given by $M(f) = \frac{d^2 f}{dt^2}$. Calculate the nullity of M, and prove that M is onto. Why doesn't this contradict the Dimension Theorem?

   The kernel of M is those polynomials whose second derivative equals zero. This is precisely $\mathbb{R}_1[t] = \{a + bt : a, b \in \mathbb{R}\}$, which is of dimension 2. Hence the nullity of M is 2. Yet M is onto, because every polynomial is twice-integrable (i.e. given any polynomial f, there is at least one polynomial g such that $\frac{d^2 g}{dt^2} = f$). This does not violate the Dimension Theorem because $\mathbb{R}[t]$ is infinite-dimensional.